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Similarity solution for a spherical radiation-driven shock wave

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A similarity solution has been obtained for a radiation-driven shock wave. Radiation propagating radially inwards is completely absorbed in the shock layer of a spherical, expanding shock wave. For a strong shock wave and a constant power input a similarity solution is obtained. It is found that the radial position of the shock wave $r_s \sim t^{\frac{3}{5}}$. The shock wave propagates as an overdriven detonation. The jump conditions and complete flow field are obtained.

1. Introduction

A similarity solution for the spherical blast wave has been given by Sedov (1946*a*, *b*) and Taylor (1950*b*). An amount of energy E_0 is introduced instantaneously at the origin; subsequently a strong, spherical shock wave propagates into the surrounding gas. The shock wave decays in such a manner that the energy associated with the spherical flow field is equal to the initial energy E_0. The similarity solution remains valid as long as the strong shock approximation is applicable across the shock wave.

A similarity solution has also been obtained for a strong spherical detonation (Zeldovich 1942; Taylor 1950*a*). In this case the spherical detonation wave propagates at a constant speed, the Chapman–Jouguet speed. An important feature of this solution is the presence of a central region in which there is no flow.

In this paper a similarity solution is obtained for a spherical, radiation-driven shock wave. It is assumed that radiation propagates radially inwards with a constant power P. This radiation propagates through a gas of density ρ_0 which is transparent. A strong spherical shock wave is generated at the origin and the incident radiation is absorbed within the shock layer. The steady, one-dimensional propagation of this type of shock wave has been studied by Ramsden & Savic (1964) and Raizer (1965). The solution obtained by these authors is used as a jump condition in this paper. The behaviour of the wave is similar to that of a detonation, a Chapman–Jouguet condition is found. The speed of the spherical, radiation-driven shock wave is a function of time because the flux density of radiation at the shock front decreases with time. The jump conditions across the expanding shock wave are matched to a particle–isentropic flow behind the shock wave. The energy associated with this spherical flow field is equal to the energy that has been absorbed in the shock layer. The similarity solution remains valid as long as the strong shock approximation is appropriate across the shock layer. The problem is illustrated in figure 1.

A spherical, radiation-driven shock wave could be generated by a number of lasers focused on a common point so that the entire solid angle is uniformly filled with radiation as proposed by Daiber, Hertzberg & Wittliff (1966). The initial wave could be generated by the spontaneous breakdown of the gas or by the use of a small, energy absorbing particle.

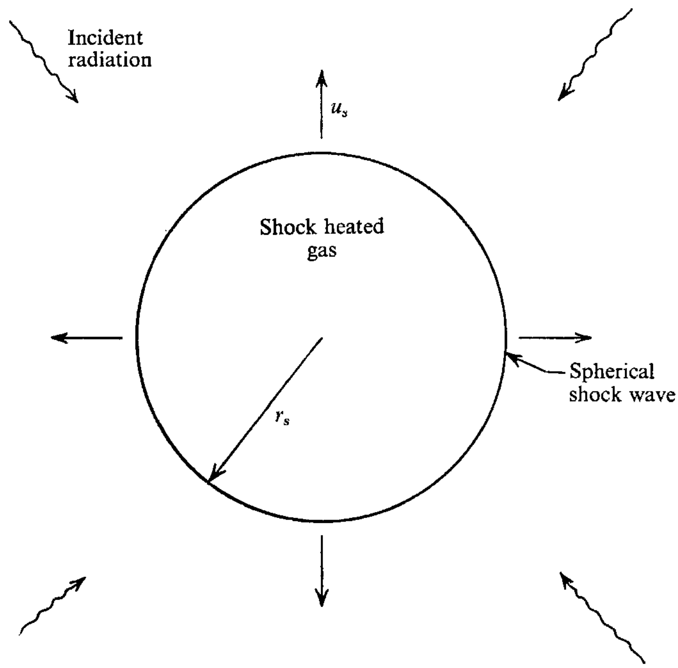

FIGURE 1. Illustration of the problem.

2. Similarity analysis

The problem we consider has complete spherical symmetry. The incident radiation is propagating radially inwards. The total flux of radiation at any radius is P and is independent of time; the flux per unit area at radius r is $P/4\pi r^2$. A strong spherical shock wave is propagating outward and is located at $r = r_s$. The incident radiation is completely absorbed within the shock layer. Reradiation from the shock layer is not considered. The gas into which the shock wave is propagating has a density ρ_0 and is optically thin so that there is no interaction with the incident radiation. The radiation does not penetrate through the shock layer which will be treated as a discontinuity; and the flow of the gas processed by the shock wave is particle isentropic.

For the expansion of a strong shock wave the governing parameters are the power of the incident radiation P and the ambient density ρ_0. From dimensional analysis the radial position of the shock wave as a function of time is given by

$$r_s \sim \left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} t^{\frac{3}{5}} \tag{1}$$

and the velocity of the shock wave U_s is given by

$$U_s \sim \left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} t^{-\frac{2}{5}}. \tag{2}$$

The dependences of the shock position and velocity on time are similar to those for a spherical blast wave except that the powers are $\frac{3}{5}$ and $-\frac{2}{5}$ instead of $\frac{2}{5}$ and $-\frac{3}{5}$.

3. Equations

The shock layer including the region of energy absorption will be treated as a discontinuity and the appropriate jump conditions are given. It is assumed that the strong shock approximation is applicable to the expanding shock wave. The energy absorbed per unit area and time in the shock layer is $P/4\pi r^2$. The equations for conservation of mass, momentum, and energy across the shock layer are

$$\rho_0 U_s = \rho_1(U_s - u_1), \tag{3}$$

$$\rho_0 U_s^2 = \rho_1(U_s - u_1)^2 + p_1, \tag{4}$$

$$\tfrac{1}{2}U_s^2 + \frac{P}{4\pi r_s^2 \rho_0 U_s} = \tfrac{1}{2}(U_s - u_1)^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} \tag{5}$$

in a laboratory co-ordinate system. The subscript 1 denotes conditions immediately downstream of the shock layer. The pressure terms on the left side of (4) and (5) have been dropped because of the strong shock approximation. Equations (3) to (5) have been previously given by Raizer (1965). The resulting Hugoniot is similar to that for a detonation. A Chapman–Jouguet condition is found and it is concluded that a wave that is not overdriven will propagate at the Chapman–Jouguet speed.

There is no radiation behind the expanding shock wave and the flow is particle isentropic. The governing equations for conservation of mass, momentum and energy are

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{7}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial r}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{8}$$

where it is assumed that the ideal gas approximation is valid with γ the ratio of specific heats.

4. Similarity solution

Following the general similarity analysis of Sedov (1959) we introduce the following dimensionless variables

$$V = \frac{ut}{r}, \quad X = \frac{\rho}{\gamma p}\left(\frac{r}{t}\right)^2,$$

$$R = \frac{\rho}{\rho_0}, \quad \lambda = \left(\frac{\rho_0}{P}\right)^{\frac{1}{5}} \frac{r}{t^{\frac{3}{5}}}. \tag{9}$$

The position of the shock wave is given by

$$r_s = \lambda_s \left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} t^{\frac{3}{5}}, \tag{10}$$

where λ_s is the value of the dimensionless co-ordinate at the shock wave. And the velocity of the shock wave is obtained by taking the derivative of (10),

$$U_s = \tfrac{3}{5}\lambda_s \left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} \frac{1}{t^{\frac{2}{5}}}. \tag{11}$$

These results are consistent with the dimensional analysis given in (1) and (2).

Substitution of the dimensionless variables into the equations for conservation of mass, momentum and energy across the shock layer gives

$$\tfrac{3}{5} = R_1(\tfrac{3}{5} - V_1), \tag{12}$$

$$\tfrac{9}{25} = R_1(\tfrac{3}{5} - V_1)^2 + \frac{R_1}{\gamma X_1}, \tag{13}$$

$$\frac{9}{50} + \frac{5}{12\pi\lambda_s^5} = \tfrac{1}{2}(\tfrac{3}{5} - V_1)^2 + \frac{1}{(\gamma-1)\,X_1}. \tag{14}$$

These equations are solved to give the dimensionless velocity, density, and reciprocal temperature immediately downstream of the shock layer in terms of the dimensionless position of the shock wave λ_s,

$$V_1 = \frac{3}{5(\gamma+1)} + \left\{\left[\frac{3}{5(\gamma+1)}\right]^2 - \frac{5}{6\pi\lambda_s^5}\left[\frac{\gamma-1}{\gamma+1}\right]\right\}^{\frac{1}{2}}, \tag{15}$$

$$R_1 = \left\{\frac{\gamma}{\gamma+1} - \left[\left(\frac{1}{\gamma+1}\right)^2 - \frac{125}{54\pi\lambda_s^5}\left(\frac{\gamma-1}{\gamma+1}\right)\right]^{\frac{1}{2}}\right\}^{-1}, \tag{16}$$

$$\begin{aligned} X_1 = \frac{1}{\gamma}\left[\frac{3}{5(\gamma+1)} + \left\{\left[\frac{3}{5(\gamma+1)}\right]^2 - \frac{5}{6\pi\lambda_s^5}\left[\frac{\gamma-1}{\gamma+1}\right]\right\}^{\frac{1}{2}}\right]^{-1} \\ \times\left[\frac{3\gamma}{5(\gamma+1)} - \left\{\left[\frac{3}{5(\gamma+1)}\right]^2 - \frac{5}{6\pi\lambda_s^5}\left[\frac{\gamma-1}{\gamma+1}\right]\right\}^{\frac{1}{2}}\right]^{-1}. \end{aligned} \tag{17}$$

These equations give the Hugoniot curve in terms of the parameter λ_s.

The Chapman–Jouguet condition requires that the downstream flow be sonic in shock-fixed co-ordinates, that is

$$U_{s\mathrm{CJ}} - u_{1\mathrm{CJ}} = a_{1\mathrm{CJ}} = \left(\frac{\gamma p_{1\mathrm{CJ}}}{\rho_{1\mathrm{CJ}}}\right)^{\frac{1}{2}}. \tag{18}$$

Introducing the dimensionless variables the Chapman–Jouguet condition becomes

$$\tfrac{3}{5} - V_{1\mathrm{CJ}} = \frac{1}{X_{1\mathrm{CJ}}^{\frac{1}{2}}}. \tag{19}$$

If the Chapman–Jouguet condition is satisfied the dimensionless position of the shock wave is

$$\lambda_{s\mathrm{CJ}} = \left[\frac{125}{54\pi}(\gamma^2 - 1)\right]^{\frac{1}{5}}, \tag{20}$$

and the corresponding values of the dimensionless variables behind the shock wave are

$$V_{1\mathrm{CJ}} = \frac{3}{5(\gamma+1)}, \quad R_{1\mathrm{CJ}} = \frac{\gamma+1}{\gamma}, \quad X_{1\mathrm{CJ}} = \left[\frac{5(\gamma+1)}{3\gamma}\right]^2. \tag{21}$$

If the shock wave propagates at the Chapman–Jouguet speed these relations are valid, however, this remains to be shown.

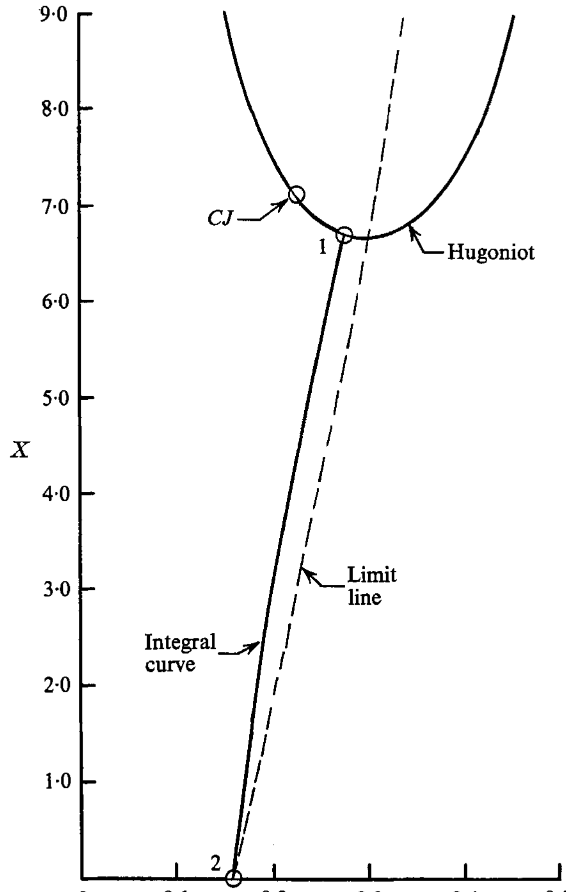

FIGURE 2. Phase plane solution, the integral curve gives the solution from the shock layer (1) to the centre of the flow (2). The Chapman–Jouguet point (CJ) is shown for comparison.

In order to analyze the flow behind the shock wave the non-dimensional variables are substituted into (6) to (8) with the result, after some manipulation,

$$\frac{d\ln X}{dV} = \frac{(\gamma-1)V(V-1)(V-\frac{3}{5})X - [2(V-1)+3(\gamma-1)V][V-\frac{3}{5}]^2X + 2(V-1) + [4(\gamma-1)/5\gamma]}{[V-\frac{3}{5}][V(V-1)(V-\frac{3}{5})X + (4/5\gamma) - 3V]}, \tag{22}$$

$$\frac{d\ln\lambda}{dV} = \frac{1-(V-\frac{3}{5})^2X}{V(V-1)(V-\frac{3}{5})X+(4/5\gamma)-3V}, \tag{23}$$

$$\frac{d\ln R}{d\ln\lambda} = -\frac{3V}{V-\frac{3}{5}} - \frac{V(V-1)(V-\frac{3}{5})X+(4/5\gamma)-3V}{(V-\frac{3}{5})[1-(V-\frac{3}{5})^2]X}. \tag{24}$$

These equations are similar to those given by Sedov (1959) for the spherical blast wave problem.

An integral curve in the X, V phase plane can be obtained by integrating (22). Conditions immediately behind the shock layer must lie on the Hugoniot curve obtained from (15) to (17). For $\gamma = \frac{5}{3}$ this Hugoniot in the X, V plane is given in figure 2. The Chapman–Jouguet point on the Hugoniot curve as obtained from (21) is also given in figure 2. The dimensionless position of a Chapman–Jouguet

V	X	λ	R
0·160	0	0	0
0·170	0·947	0·525	0·449
0·180	1·747	0·641	0·588
0·190	2·438	0·717	0·701
0·200	3·047	0·775	0·805
0.210	3·597	0·823	0·907
0·220	4·105	0·866	1·013
0·230	4·584	0·904	1·126
0·240	5·047	0·940	1·249
0·250	5·504	0·975	1·387
0·260	5·963	1·010	1·544
0·270	6·433	1·025	1·727
0·276	6·722	1·067	1·852

TABLE 1

shock wave is $\lambda_s = 1{\cdot}055$. The singularity in the phase plane corresponding to the centre of the sphere is at $X = 0$ and $V = \frac{4}{25}$. This singularity is a saddle point and the integral curve leaving it is obtained by expanding (22) about the singularity. A numerical integration of (22) has been carried out to complete the integral curve to the intersection with the Hugoniot. This integral curve is shown in figure 2 and is tabulated in table 1. The intersection *does not* occur at the Chapman–Jouguet point. The shock behaves like an overdriven detonation. The value of the dimensionless position of the shock wave is $\lambda_s = 1{\cdot}067$. The intersection gives $V_1 = 0{\cdot}276$ and $X_1 = 6{\cdot}722$.

A physical explanation for the overdriven behaviour of the shock wave can be given. We have shown that the shock speed in this problem is not constant, see (2). In order for the shock wave to slow down it is necessary that weak disturbances behind the shock wave be able to catch up with the shock wave. This would not be the case for a Chapman–Jouguet wave since the flow speed behind the wave would be sonic in shock fixed co-ordinates. For an overdriven wave the downstream Mach number in shock fixed co-ordinates is subsonic and weak waves can reach the shock layer.

Using the integral solution of (22), (23) is integrated numerically to give the dependence of the dimensionless co-ordinate λ on the dimensionless velocity V. And then the dimensionless density R is obtained by a numerical integration of (24). The value of the dimensionless density just downstream of the shock layer is $R_1 = 1{\cdot}852$. The results are tabulated in table 1.

5. Results

The position of the shock wave as a function of time is obtained by substituting the value of the dimensionless shock position, λ_s, as obtained from the numerical solution into (10) with the result

$$r_s = 1{\cdot}067\left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \tag{25}$$

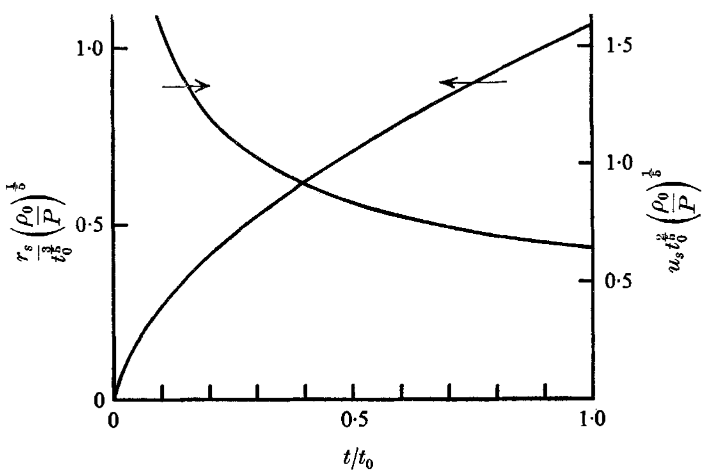

FIGURE 3. Dependence of shock position and velocity on time, t_0 is a characteristic time.

and the shock velocity as a function of time from (11) is

$$U_s = 0{\cdot}640\left(\frac{P}{\rho_0}\right)^{\frac{1}{5}} t^{-\frac{3}{5}}. \tag{26}$$

These results are illustrated in figure 3 in terms of a reference time t_0.

The dimensionless variables X, λ and R have been obtained in terms of V. From this result it is straightforward to obtain the dimensional variables u, ρ, p and T. The radial dependence of the variables is illustrated in figure 4. The dependent variables are normalized using their values immediately downstream of the shock layer and the radial distance is normalized by the radial position of the shock wave. It is seen that the velocity and density go to zero at the centre while the temperature approaches infinity. The pressure remains finite at the centre.

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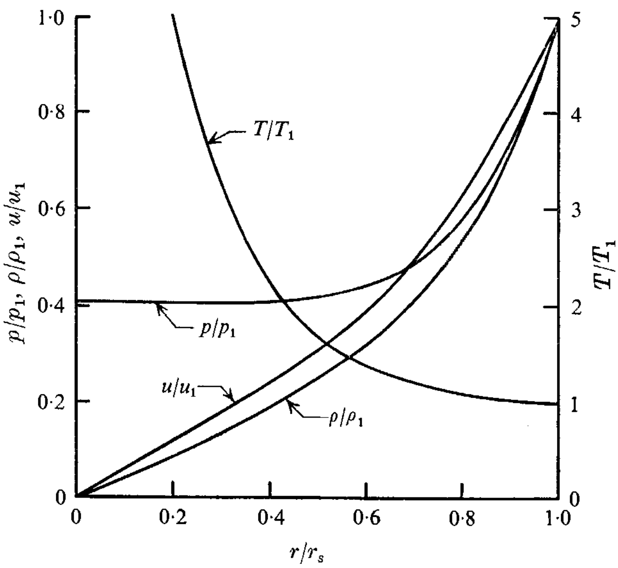

FIGURE 4. Dependence of velocity, density, pressure and temperature on radius.

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Note added in proof: The authors' attention has been called to similar work by Champetier *et al.* (1968).